CONSTRUCTIVE DECOMPOSITION OF A FUNCTION OF TWO VARIABLES AS A SUM OF FUNCTIONS OF ONE VARIABLE

EVA TRENKLEROVÁ

Abstract. Given a compact set K in the plane, which does not contain any triple of points forming a vertical and a horizontal segment, and a map $f \in C(K)$, we give a construction of functions $g, h \in C(\mathbb{R})$ such that $f(x, y) = g(x) + h(y)$ for all $(x, y) \in K$. This provides a constructive proof of a part of Sternfeld's theorem on basic embeddings in the plane. In our proof the set K is approximated by a finite set of points.

1. Introduction

An embedding $\varphi\colon K \to \mathbb{R}^k$ of a compactum (compact metric space) K in the k-dimensional Euclidean space $\mathbb{R}^k$ is called a *basic embedding* provided that for each continuous real-valued function $f \in C(K)$, there exist continuous real-valued functions of one real variable $g_1, \dots, g_k \in C(\mathbb{R})$ such that $f(x_1, \dots, x_k) = g_1(x_1) + \dots + g_k(x_k)$ for all points $(x_1, \dots, x_k) \in \varphi(K)$. We also say, that the set $\varphi(K)$ is *basically embedded* in $\mathbb{R}^k$.

The question of the existence of basic embeddings was already implicitly contained in Hilbert's 13th problem [Hil00]: Hilbert conjectured that not all continuous functions of three variables were expressible as sums and superpositions of continuous functions of a smaller number of variables.

Ostrand [Ost65] proved that each n-dimensional compactum can be basically embedded in $\mathbb{R}^{2n+1}$ for $n \geq 1$. His result is an easy generalization of results of Arnold [Arn57, Arn59] and Kolmogorov [Kol56, Kol57].

Sternfeld [Ste85] proved that the parameter $2n + 1$ is the best possible in a very strong sense: namely, that no n-dimensional compactum can be basically embedded in $\mathbb{R}^{2n}$ for $n \geq 2$. Ostrand's and Sternfeld's results thus characterize compacta basically embeddable in $\mathbb{R}^k$ for $k \geq 3$. Basic embeddability in the real line is trivially equivalent to embeddability. The remaining problem of the characterization of compacta basically embedded in $\mathbb{R}^2$ was already raised by Arnold [Arn58] and solved by Sternfeld [Ste89]:

Theorem 1.1 (Sternfeld). *Let K be a compactum and let $\varphi\colon K \to \mathbb{R}^k$ be an embedding. Then*

(B) φ is a basic embedding

if and only if

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(A) there exists an $m \in \mathbb{N}$ such that the set $\varphi(K)$ does not contain an array of length m.

Definition 1.1. An *array* is a sequence of points $\{z_i\}_{i \in I}$ in the plane, where $I = \{1, 2, \dots, m\}$ or $I = \mathbb{N}$, such that for each i:

- $z_i \neq z_{i+1}$ and $[z_i; z_{i+1}]$ is a segment parallel to one of the coordinate axes and
- the segments $[z_i; z_{i+1}]$ and $[z_{i+1}; z_{i+2}]$ are mutually orthogonal.

If $I = \{1, 2, \dots, m\}$ then the length of the array is $m - 1$.

Using the geometric description (A), Skopenkov [Sko95] gave a characterization of continua basically embeddable in the plane by means of forbidden subsets resembling Kuratowski's characterization of planar graphs. In a similar way Kurlin [Kur00] characterized finite graphs basically embeddable in $\mathbb{R} \times T_n$, where T_n is a star with n-rays. Repovš and Željko [RŽ06] proved a result concerning the smoothness of functions in a basic embedding in the plane.

Sternfeld's proof of the equivalence (A) $\Leftrightarrow$ (B) is not direct but uses a reduction to linear operators. In particular it is not constructive. It is therefore desirable to find a straightforward, constructive proof which will consequently provide an elementary proof of Skopenkov's and Kurlin's characterizations. A constructive proof of (B) $\Rightarrow$ (A) is given in [MKT03].

In this paper we give such an elementary construction, thus proving the implication (A) $\Rightarrow$ (B) provided that $m = 2$:

Theorem 1.2. *Let $\varphi\colon K \to \mathbb{R}^2$ be an embedding of a compactum K in the plane such that the set $\varphi(K)$ does not contain an array of length two. Then for every function $f \in C(\varphi(K))$ there exist functions $g, h \in C(\mathbb{R})$ such that $f(x, y) = g(x) + h(y)$ for all points $(x, y) \in \varphi(K)$.*

The main part of our proof consists in finding an approximate decomposition of a given function f as $g + h$. The functions g, h are defined on a finite approximation V^n of $\varphi(K)$. Then they are linearly extended to $\mathbb{R}$. Apart from two steps, where we asset the existence of certain constants, this part of the proof is constructive. The existence of an exact decomposition follows by an elementary iterative procedure.

Until now, no constructive decomposition of f as $g + h$ on compacta in the plane satisfying (A) of Theorem 1.1 has been found, not even in the simplest case, when the compactum satisfies (A) with $m = 2$.

Our result resembles representation theorems of Arnold [Arn57, Arn59], Kolmogorov [Kol56, Kol57] and Ostrand [Ost65]. The proofs are similar in that we also construct a sequence of finite families of squares. But different from these proofs, where the squares (or cubes in higher dimensions) are connected only with the dimension of the set in question, here the squares mimic the property that the set does not contain an array of length two.

In the paper [RŽ06] the authors give the decomposition for finite graphs basically embedded in the plane: according to the results of [Sko95, CRS98], a finite graph can be basically embedded in the plane if and only if it can be embedded in a special graph R_n for some n. The authors of [RŽ06] inductively define an embedding $\varphi\colon R_n \to \mathbb{R}^2$. For a given function $f \in C(\varphi(R_n))$ they define the maps $g, h \in C(\mathbb{R})$ inductively again, starting from a well chosen subset of $\varphi(R_n)$.

Although the sets we are dealing with do not contain arrays of length two, they can be still "arbitrarily bad". In particular we are not able to choose a suitable

subset to start the construction on. Thus it turns out, that even if a set $\varphi(K) \subseteq \mathbb{R}^2$ satisfies the simplest version of condition (A), a constructive decomposition of a function $f \in C(\varphi(K))$ is a non-trivial problem. We believe, that the proof can be modified to obtain a constructive proof of the implication (A) $\Rightarrow$ (B) for an arbitrary $m \in \mathbb{N}$.

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2. Notation and conventions

Throughout the text we fix an embedding $\varphi\colon K \to \mathbb{R}^2$ of a compactum K in the plane such that the set $\varphi(K)$ does not contain an array of length two. For simplicity of notation we identify the set K and its homeomorphic image $\varphi(K)$ and we speak about a set $K \subseteq \mathbb{R}^2$. Let f be from $C(K)$ and $\varepsilon > 0$ be the desired approximation constant. Since f is continuous on the compact set K, it is uniformly continuous there. Therefore, there exists a positive real $\delta = \delta(K, f, \varepsilon) > 0$ such that for all points $z, z' \in K$ if $|z - z'| < \delta$ then $|f(z) - f(z')| < \varepsilon$. We fix this δ as well.

The distance in $\mathbb{R}^2$ is defined as $|(x, y) - (x', y')| = \max\{|x - x'|, |y - y'|\}$ for $(x, y), (x', y') \in \mathbb{R}^2$. By $p, q\colon \mathbb{R}^2 \to \mathbb{R}$ we denote the vertical and horizontal orthogonal projections: $p(x, y) = x$, $q(x, y) = y$.

3. Idea of the proof and the main statements

Our proof of Theorem 1.2 mimics the following construction of the functions g, h which works for certain types of sets K (for example graphs, considered in [RŽ06]). Denote by K_x the set of all points $(x, y) \in K$ which have a neighbor in the vertical direction in K, i.e. $K_x = \{(x, y) \in K | \exists (x, y') \in K, y \neq y'\}$. Similarly define K_y. Assume that both sets K_x and K_y are closed.

Since K does not contain an array of length two, the functions p and q are injective on K_y and K_x, respectively, and the sets K_x and K_y are disjoint. For each point $x \in p(K_y)$ let $g(x) = f(x, p^{-1}(x))$ and for each point $x \in p(K_x)$ let $g(x) = 0$. Extend g continuously to $\mathbb{R}$. The function h is defined in the following way: for each point $y \in q(K)$ pick an arbitrary point $(x, y) \in K$ and let $h(y) = f(x, y) - g(x)$. It is easily seen that h is continuous. We extend h to $\mathbb{R}$.

We have defined the functions g and h first on the sets K_x and K_y. In general these sets are not closed, and the set K can be so "bad" that we cannot find a suitable set to begin the definitions.

The main part of the proof consists of three steps.

Step 1. For each n we construct the set V^n approximating K. Consider the lattice $(i/2^n, j/2^n)$ with $i, j \in \mathbb{Z}$. For each square $[i/2^n; (i+1)/2^n) \times [j/2^n; (j+1)/2^n)$ which intersects the set K we choose one point from the intersection of this square with the set K. The set V^n consists of all the chosen points.

We shall call a segment $[(u_1, v_1); (u_2, v_2)]$ given by a pair of points (u_1, v_1), $(u_2, v_2) \in V^n$ *almost vertical in* V^n if $|u_1 - u_2| < 2/2^n$ and *almost horizontal in* V^n if $|v_1 - v_2| < 2/2^n$.

Points which are the ends of almost vertical or horizontal segments are "near" to each other in the vertical or horizontal direction, respectively.

Arrays in the set K are "deformed" in the approximating finite set V^n so we generalize the notion of an array, in the following way.

Definition 3.1. A sequence of pairwise different points $\{w_1, w_2, \dots, w_m\}$ from the set V^n is said to form an *almost array in* V^n if each pair of consecutive points w_i, w_{i+1} forms an almost vertical or horizontal segment in V^n. The *length* of the almost array is defined to be $m-1$.

Step 2. In this step we define approximations G^n and H^n of g and h on the set V^n. This step contains the major part of the proof and consists of proving three statements.

If the distance of two points from V^n is smaller than δ then the difference of f between them is bounded by ε. As we are approximating up to ε, such points are "almost the same" for us.

Definition 3.2. A segment $[z_1; z_2]$ with $z_1, z_2 \in \mathbb{R}^2$ is said to be *long* if $|z_1 - z_2| \geq \delta$ and it is said to be *short* if $|z_1 - z_2| < \delta$.

The set of all points from V^n which are the ends of the long almost vertical segments in V^n is the analogue of the set K_x and the set of all points from V^n which are the ends of the long almost horizontal segments in V^n is the analogue of the set K_y.

Theorem 3.3. *There exists an n_1 such that for all $n \geq n_1$ a function $G^n : p(V^n) \to \mathbb{R}$ satisfying the following requirements exists.*

(1) (a) $|G^n(u_1) - G^n(u_2)| \leq 3\varepsilon$ *for each short segment* $[(u_1, v_1); (u_2, v_2)]$ *in* V^n
   (b) $|G^n(u) - f(u, v)| \leq 2\varepsilon$ *for each* $(u, v) \in V^n$ *which is the end of a long almost horizontal segment in* V^n
   (c) $|G^n(u)| \leq \varepsilon$ *for each* $(u, v) \in V^n$ *which is the end of a long almost vertical segment in* V^n

(2) $||G^n|| \leq ||f||$.

The function H^n is constructed using G^n from Theorem 3.3:

Theorem 3.4. *There exists an n_2 such that for all $n \geq n_2$ functions $G^n : p(V^n) \to \mathbb{R}$ and $H^n : q(V^n) \to \mathbb{R}$ satisfying the following requirements exist.*

(1) (a) $|f(u, v) - G^n(u) - H^n(v)| \leq 4\varepsilon$ *for each* $(u, v) \in V^n$
   (b) $|G^n(u_1) - G^n(u_2)| \leq 3\varepsilon$ *for each segment* $[(u_1, v_1); (u_2, v_2)]$ *which is almost vertical in* V^n
   (c) $|H^n(v_1) - H^n(v_2)| \leq 12\varepsilon$ *for each segment* $[(u_1, v_1); (u_2, v_2)]$ *which is almost horizontal in* V^n

(2) $||G^n|| \leq ||f||$, $||H^n|| \leq 2||f||$.

In order to construct the function G^n from Theorem 3.3 we need the following lemma. Its proof is based on the fact that K is compact and does not contain an array of length 2. It will be used with $l = [||f||/\varepsilon]$.

Lemma 3.5. *For each $l \in \mathbb{N}$ there exists an n_0 such that for all $n \geq n_0$ the following holds: if $\{w_1, \dots, w_k\}$ is an almost array in V^n and w_1 is the end of a long almost vertical segment in V^n and w_k is the end of a long almost horizontal segment in V^n, then the length of the almost array is at least l, i.e. $k - 1 \geq l$.*

Step 3. The functions g, h are obtained by linear extensions of the functions G^n, H^n.

Theorem 3.6. *There exists n_3 such that for all $n \geq n_3$ functions $g, h \in C(\mathbb{R})$ satisfying the following requirements exist. such that*

(1) *$|f(x,y) - g(x) - h(y)| \leq 20\varepsilon$ for all points $(x,y) \in K$*
(2) *$||g|| \leq ||f||$, $||h|| \leq 2||f||$.*

4. Proof of the main statement

Proof of Theorem 1.2. Assuming that the above three steps have been accomplished, the statement follows immediately from Theorem 3.6 and Theorem 4.1 below. □

To make the proof clearer though, we explicitly describe our construction.

According to Lemma 3.5 there exists an n_0 such that for all $n \geq n_0$ every almost array in V^n starting in a long almost vertical segment and ending in a long almost horizontal segment has length at least $[||f||/\varepsilon]$. We define a constant $N = \max\{n_0, -[\log_2 \delta]\}$ and let the lower bounds n_1, n_2 and n_3 from Theorems 3.3, 3.4 and 3.6 be all equal to N. We take an arbitrary $n \geq N$.

We construct V^n and the corresponding function $G^n : p(V^n) \to \mathbb{R}$ from Theorem 3.3 which approximates g. Using G^n, we define the function $H^n : q(V^n) \to \mathbb{R}$ from Theorem 3.4 thus obtaining the approximate of h. The functions G^n and H^n are extended as piecewise linear functions on $\mathbb{R}$ thus obtaining the functions g and h from Theorem 3.6. Applying Theorem 4.1 below we obtain the exact decomposition.

Theorem 4.1 (implication (b) $\Rightarrow$ (c) of Theorem 4.13 in [Rud91])**.** *Let $X \subseteq \mathbb{R}^2$ be an arbitrary compact subset of the plane. Assume that there exists a positive integer $k \in \mathbb{N}$ such that for each function $f \in C(X)$ and each positive real $\varepsilon > 0$ there exist functions $g', h' \in C(\mathbb{R})$ such that*

(1) *$|f(x,y) - g'(x) - h'(y)| \leq \varepsilon$ for all points $(x,y) \in X$*
(2) *$||g'|| \leq k||f||$, $||h'|| \leq k||f||$.*

Then there exist functions $g, h \in C(\mathbb{R})$ such that $f(x,y) = g(x) + h(y)$ for all points $(x,y) \in X$.

5. Proofs of the statements

Let us give the proofs of the statements in the order in which we use them to prove Theorem 1.2.

Proof of Lemma 3.5. First, let us note the following. Let $\{[w_1^n; w_2^n]\}_{n=1}^{\infty}$ be a sequence where each $[w_1^n; w_2^n]$ is an almost vertical or an almost horizontal segment in V^n. Then, since K is compact, there is a subsequence $\{[w_1^{m_n}; w_2^{m_n}]\}_{m_n}$ such that both $w_1^{m_n} \to w_1 \in K$ and $w_2^{m_n} \to w_2 \in K$ as $n \to \infty$. Evidently, either $w_1 = w_2$ or $[w_1; w_2]$ is a segment parallel to one of the coordinate axes. Moreover, if $|p(w_1^{m_n}) - p(w_2^{m_n})| \geq \delta$ for each n then $|p(w_1) - p(w_2)| \geq \delta$ and $[w_1; w_2]$ is a segment parallel to the x axis. Similarly for the projection q.

Assuming that the statement is not true, we will show that K contains an array of length two. So, assume that for some l_0 there exists an increasing sequence $\{m_n\}_{n=1}^{\infty}$ of integers such that each set V^{m_n} contains an almost array $\{w_1^{m_n}, w_2^{m_n}, \ldots, w_{k^{m_n}}^{m_n}\}$ as in the statement, but its length $k^{m_n} - 1$ is smaller than l_0, so $k^{m_n} \leq l_0$. This implies that infinitely many of the numbers k^{m_n} are the same. Without loss of

generality we may assume that $k^{m_n} = l_0$ for all n. For each i, the point w_1^i is the end of a long almost vertical segment in V^n denoted by $[w_0^i; w_1^i]$ and the point $w_{l_0}^i$ is the end of a long almost vertical segment in V^n, denoted by $[w_{l_0}^i; w_{l_0+1}^i]$.

It follows, that there exist limit points $w_0, w_1, \ldots, w_{l_0+1} \in K$ such that either $w_i = w_{i+1}$ or $[w_i; w_{i+1}]$ is a segment parallel to one of the coordinate axes for each i. In particular, $[w_0; w_1]$ is a vertical segment and $[w_{l_0}; w_{l_0+1}]$ is a horizontal segment. Therefore, the set $\{w_0, w_1, \ldots, w_{l_0+1}\} \subseteq K$ contains an array of length two. $\square$

Proof of Theorem 3.3. Denote $F = [||f||/\varepsilon]$. Let n_1 be equal to $n_0 = n(l)$ from Lemma 3.5 which corresponds to $l = F$. Take an arbitrary $n \geq n_1$.

First we define a function $\gamma \colon V^n \to \mathbb{R}$ such that

- (i) (a) $|\gamma(w_1) - \gamma(w_2)| \leq \varepsilon$ for each short segment $[w_1; w_2]$
  - (b) $|f(w) - \gamma(w)| \leq \varepsilon$ for each w which is the end of a long almost horizontal segment
  - (c) $\gamma(w) = 0$ for each w which is the end of a long almost vertical segment
- (ii) $||\gamma|| \leq ||f||$.

Second, we define G^n using γ; we shall have roughly $G^n(u) = \gamma(u,v) \pm \varepsilon$ for all $(u,v) \in V^n$. To construct γ, we define two abstract graphs with vertices from V^n. They are not embedded in the plane.

Assume, that for each $i = -F, -F+1, \ldots, F$ there exists a point $w \in V^n$ such that $[f(w)/\varepsilon] = i$. If this is not the case then we add a new point $z \in \mathbb{R}^2$ to V^n for each i for which no such point exists. Formally we consider it as the end of a long almost horizontal segment, and let $f(z) = i\varepsilon$. The point is added so that its distance from each point from V^n is greater than δ.

Let V_+ be the set of all points $w \in V^n$ with $f(w) \geq 0$ with one vertex $\mathbf{w}_+$ added:

$$V_+ = \{w \in V^n \mid f(w) \geq 0\} \cup \{\mathbf{w}_+\}.$$

We define $f(\mathbf{w}_+) = (F+1)\varepsilon$. Let V_- be the set of all points $w \in V^n$ with $f(w) < 0$ with one vertex $\mathbf{w}_-$ added:

$$V_- = \{w \in V^n \mid f(w) < 0\} \cup \{\mathbf{w}_-\}$$

and let $f(\mathbf{w}_-) = (-F-1)\varepsilon$.

The edge set $E(V_+)$ consists of edges

- $w_1 w_2$ where $[w_1; w_2]$ is a short segment
- $w_1 w_2$ where both w_1 and w_2 are the ends of long almost horizontal segments in V^n and $[f(w_1)/\varepsilon] - [f(w_2)/\varepsilon] = 1$
- $\mathbf{w}_+ w$ where w is the end of a long almost horizontal segment in V^n and $[f(w)/\varepsilon] = F$.

Evidently

$$|f(w_1) - f(w_2)| \leq \varepsilon \tag{5.1}$$

for each edge $w_1 w_2 \in E(V_+)$. The edges $E(V_-)$ are defined analogously.

Let $d \colon V_+ \to \{0, 1, 2, \ldots\}$ be the function, assigning to each vertex which is connected to $\mathbf{w}_+$ by a path in $E(V_+)$ its distance from $\mathbf{w}_+$, and assigning to each other vertex the value 0. For each vertex $w \in V_+$ let

$$\gamma(w) = \max\{(F - d(w) + 1)\varepsilon, 0\}. \tag{5.2}$$

Analogously we define the function γ on V_-.

Let us show that the function γ satisfies (i) and (ii).

(ia) Let $[w_1; w_2]$ be a short segment in V^n. If both w_1 and w_2 are in V_+, or both w_1 and w_2 are in V_- then $|\gamma(w_1) - \gamma(w_2)| \le \varepsilon$ follows directly from the definition of γ. So, let $w_1 \in V_+$ and $w_2 \in V_-$. Using (5.1), (5.2), by induction on the distance from the vertex $\mathbf{w}_+$ and analogously, by induction on the distance from $\mathbf{w}_-$ we can show that

$$\begin{array}{rll} 0 \le \gamma(w) \le f(w) & \text{for all} & w \in V_+ \\ f(w) \le \gamma(w) \le 0 & \text{for all} & w \in V_- \end{array} \tag{5.3}$$

Since the segment $[w_1; w_2]$ is short we have $|f(w_1) - f(w_2)| < \varepsilon$. So $|\gamma(w_1) - \gamma(w_2)| = \gamma(w_1) - \gamma(w_2) \le f(w_1) - f(w_2) < \varepsilon$.

(ib) Let w be the end of a long almost horizontal segment in V^n. Let $w \in V_+$ for instance. Denote $i = [f(w)/\varepsilon]$. By definition there is a path $\mathbf{w}_+ w_F \ldots w_{i+2} w_{i+1} w$ whose edges are in $E(V_+)$. Each of its vertices w_j is the end of a long almost horizontal segment in V^n and $[f(w_j)/\varepsilon] = j$. So $d(w) \le F - i + 1$. Hence $\gamma(w) = (F - d(w) + 1)\varepsilon \ge i\varepsilon > f(w) - \varepsilon$. On the other hand, Equation (5.3) implies that $\gamma(w) \le f(w)$ and (ib) follows.

(ic) Let w be the end of a long almost vertical segment in V^n. Let $w \in V_+$ for instance. If w is not connected to $\mathbf{w}_+$ by a path then, by definition, $\gamma(w) = 0$.

So, let $\mathbf{w}_+ \ldots w$ be a path such that $d(w)$ is equal to its length. Let $w' \ldots w$ be its longest subpath containing w, such that each of it edges corresponds to a short almost vertical or almost horizontal segment in V^n. Then w' is the end of a long horizontal segment in V^n. Thus the vertices of the subpath form an almost array which satisfies the requirements of Lemma 3.5. Hence, if we denote its length by p, we have $F \le p \le d(w) - 1$. So $\gamma(w) = (F - d(w) + 1)\varepsilon < \varepsilon$. On the other hand, by the definition (5.2) of γ, we have $\gamma(w) \ge 0$. Since the values of γ are integer multiples of ε, it follows that $\gamma(w) = 0$.

Point (ii) follows directly from (5.3).

The function G^n is constructed in the following way. For each point $u \in p(V^n)$ fix an arbitrary point $(u, v) \in V^n$, and define $G^n(u) = \gamma(u, v)$. Let us show that G^n satisfies (1), (2) from Theorem 3.3.

(1a) Let $[(u_1, v_1); (u_2, v_2)]$ be a short segment in V^n. Let $G^n(u_1) = \gamma(u_1, v_1')$ and let $G^n(u_2) = \gamma(u_2, v_2')$. Denote $w_i = (u_i, v_i)$ and $w_i' = (u_i, v_i')$ for $i = 1, 2$. Since $[w_1; w_2]$ is a short segment, by (ia) we have $|\gamma(w_1) - \gamma(w_2)| \le \varepsilon$. The segment $[w_1; w_1']$ is almost vertical (in fact it is vertical). If it is short then $|\gamma(w_1) - \gamma(w_1')| \le \varepsilon$, by (ia). If it is long then $\gamma(w_1) = \gamma(w_1') = 0$, by (ic). The same is true for $[w_2; w_2']$.

If both segments $[w_1; w_1']$, $[w_2; w_2']$ are short then $|G^n(u_1) - G^n(u_2)| = |\gamma(w_1') - \gamma(w_2')| \le 3\varepsilon$. If the first one is long and the second one is short then $|\gamma(w_1) - \gamma(w_2)| = |\gamma(w_2)| \le \varepsilon$ and we have $|\gamma(w_1') - \gamma(w_2')| = |\gamma(w_2')| \le |\gamma(w_2)| + \varepsilon \le \varepsilon$. If both are long then $|\gamma(w_1') - \gamma(w_2')| = 0$.

Hence we have proved (1a). Points (1b) and (1c) are proved in a similar way. Point (2) follows directly from (ii). □

Proof of Theorem 3.4. Let n_2 be equal to n_1 from Theorem 3.3. Take an arbitrary $n \ge n_2$ and the corresponding function $G^n : p(V^n) \to \mathbb{R}$ from Theorem 3.3. Define $H^n : q(V^n) \to \mathbb{R}$ in the following way: for each $v \in q(V^n)$ fix a point $(u, v) \in V^n$ and let $H^n(v) = f(u, v) - G^n(u)$.

The arguments showing that (1) and (2) are satisfied are similar to those in the proof of Theorem 3.3. □

Proof of Theorem 3.6. Take n_2 from Theorem 3.4 and let $n_3 = \max\{n_2, -[\log_2 \delta]\}$. Then $1/2^{n_3} \leq \delta$. Let $n \geq n_3$ and take functions $G^n : p(V^n) \to \mathbb{R}$ and $H^n : q(V^n) \to \mathbb{R}$ from Theorem 3.4.

Denote the points of the p projection of the set V^n by $x_1, \ldots, x_k$ with $x_i < x_{i+1}$ for all i. Let $g(x_i) = G^n(x_i)$ for each i. On each interval $[x_i; x_{i+1}]$ such that $|x_i - x_{i+1}| < 1/2^{n-1}$ extend g linearly between the values $g(x_i)$ and $g(x_{i+1})$. If an interval $[x_i; x_{i+1}]$ is such that $|x_i - x_{i+1}| \geq 1/2^{n-1}$ then there exists an interval of the form $I = [j/2^n; (j+1)/2^n)$ such that $x_i < j/2^n < (j+1)/2^n < x_{i+1}$ and $p^{-1}(I) \cap K = \emptyset$. On the intervals $[x_i; j/2^n]$ and $[(j+1)/2^n; x_{i+1}]$ extend g as a constant, equal to $g(x_i)$ and $g(x_{i+1})$, respectively. On the intervals $(-\infty; x_1]$ and $[x_k; \infty)$ extend g as a constant as well.

Denote the points of the q projection of the set V^n by $y_1, \ldots, y_l$ with $y_j < y_{j+1}$ for all j. Let $h(y_j) = H^n(y_j)$ for each j. Extend h to $\mathbb{R}$ in a similar way as g.

Every point (x, y) from K lies in a square $S = [i/2^n; (i+1)/2^n) \times [j/2^n; (j+1)/2^n)$ and there is a point $(x_i, y_j) \in S \cap V^n$. Let $x_i \leq x$ and $y_j \leq y$ for instance. If $|x_i - x_{i+1}| < 1/2^{n-1}$ then x_i and x_{i+1} are the vertical projections of the ends of an almost vertical segment in V^n. So, by point (1b) of Theorem 3.4 we have $|G^n(x_i) - G^n(x_{i+1})| = |g(x_i) - g(x_{i+1})| \leq 3\varepsilon$. Since g is linear on the interval $[x_i; x_{i+1}]$ we have $|g(x_i) - g(x)| \leq 3\varepsilon$. If $|x_i - x_{i+1}| \geq 1/2^{n-1}$ then $g(x_i) = g(x)$. Similarly we show that $|h(y_j) - h(y)| \leq 12\varepsilon$.

By point (1a) we have $|f(x_i, y_j) - g(x_i) - h(y_j)| \leq 4\varepsilon$. Since $|(x, y) - (x_i, y_j)| < 1/2^n \leq \delta$ it follows that $|f(x, y) - f(x_i, y_j)| < \varepsilon$. Finally, $|f(x, y) - g(x) - h(y)| \leq 20\varepsilon$.

The norms of the functions g and h are bounded because of point (2) of Theorem 3.4. □

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Department of Computer Science, Faculty of Science, P.J. Šafárik University, Jesenná 5, 04001 Košice, Slovakia

E-mail address: `eva.trenklerova@upjs.sk`